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RADIATIVE ENERGY TRANSFER WITHIN A HYDROGEN PLASMA*

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Abstract—The object of this investigation is to study various methods of analyzing radiative energy transport within a plasma. For this purpose, a very simple physical system is chosen consisting of a hydrogen plasma bounded by two parallel black plates, and within the plasma there is a uniform heat source per unit volume. Limited results are also presented for a hydrogen plasma in radiative equilibrium.

Since line radiation is generally characterized by relatively large optical thicknesses, whereas continuum radiation corresponds to much smaller optical thicknesses, it is found that the optically thin and optically thick limits have little practical utility, in that both line and continuum radiation will rarely be either thin or thick simultaneously. This large variation in spectral optical thickness further appears to limit the utility of a nongray differential approximation, by which the radiative flux is described through a second-order differential equation.

It is shown that line radiation can have a significant effect upon radiative energy transport within the plasma for moderate pressures. At higher pressures, however, the lines become opaque and no longer contribute to the radiative transport process. In either case, lines appear to have little influence upon radiation passing through the plasma (as opposed to radiative transport within the plasma), since the lines encompass only a small fraction of the spectrum.

NOMENCLATURE

a, constant appearing in equation (3);
b, constant appearing in equation (3);
e_{ω}, Planck's function;
L, plate spacing;
P_H, partial pressure of atomic hydrogen;
P, total pressure;
q_R, radiative heat flux;
Q, heat source per unit volume;
T, absolute temperature;
T_c, centerline temperature;
u, pressure path length, $u = P_H y$;
u_0, total pressure path length, $u_0 = P_H L$;
y, physical coordinate;
κ_{LP}, linear Planck mean absorption coefficient [cm^{-1}];
κ_R, Rosseland mean absorption coefficient [cm^{-1}];
κ_{ω}, spectral absorption coefficient [cm^{-1}];
λ, thermal conductivity;
ε, modified emissivity;
σ, Stefan–Boltzmann constant;
ω, wave number [cm^{-1}].

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INTRODUCTION

The purpose of the present paper is to investigate methods of analyzing radiative energy transfer within a high-temperature gas, and the specific case of a hydrogen plasma is considered. Radiative transfer analyses within absorbing–emitting gases characteristically involve integral or integrodifferential equations as the applicable energy equation, since the radiative flux posses-es an integral formulation. If the assumption of a gray gas is employed, the kernel function for the integrals appearing in the radiative flux equation is the second exponential integral, $E_2(t)$. The kernel function for real (nongray) gases is much more complicated, since it depends upon the specific gas as well as the state of the gas.

Nongray kernel functions have seen use in atmospheric studies, and a survey of these applications is given by Goody [1]. The appropriate kernel function is the spacial derivative of either the emissivity or the modified emissivity of the gas [1]. The formulation of the radiative flux in terms of the derivative of the gas emissivity is also discussed by Penner [2].

In the preceding applications, the nongray kernel function has been used to describe the radiation from a gas volume, assuming a known temperature distribution. The first application of a nongray kernel function to the solution of the energy equation locally throughout the gas appears to be in the work of Gille and Goody [3]. This involved an analysis of combined infrared radiation and thermal conduction within ammonia. It was shown that for linearized radiation (small temperature differences), the appropriate kernel function is the spacial derivative of the modified emissivity. In contrast to the emissivity, for which Planck's function is the weighting factor, the modified emissivity employs the derivative of Planck's function with respect to temperature as the weighting factor.

A comprehensive discussion of the role of both the emissivity and the modified emissivity in the formulation of nongray kernel functions is given by Wang [4]. The nongray kernel function for linearized radiation has also been discussed by Baldwin [5], and by Gilles, Cogley and Vincenti [6]. In addition to the work of Gille and Goody [3], application of the nongray kernel formulation to infrared radiative transfer, for which the spectral absorption coefficient corresponds to discrete vibration–rotation bands, may be found in references [7–11].

For plasma radiation, the spectral absorption coefficient results from free–free, bound–free, and bound–bound electronic transitions. The free–free and bound–free transitions give rise to a continuum contribution to the absorption coefficient, while the bound–bound transitions result in a contribution which consists of discrete spectral lines.

In the present investigation, radiative energy transfer within a hydrogen plasma is considered. Since the absorption coefficient is non-zero throughout the spectrum, then, unlike infrared radiation [8,9], the Rosseland equation is the appropriate asymptotic limit for large optical thicknesses. The illustrative analytical model is that of a gas bounded by two parallel plates and within which there is a uniform heat source. It should be emphasized that this model is not intended to be a physically realistic system, but it is used solely to illustrate the radiative transport process for a hydrogen plasma. Limited results are also given for the case of radiative equilibrium. Linearized radiation and local thermodynamic equilibrium are assumed throughout.

Use will be made of the radiative transport quantities for a hydrogen plasma recently reported in reference [12]. These quantities include the modified emissivity and its first derivative, the linear Planck mean absorption coefficient, and the Rosseland mean absorption coefficient. The spectral absorption coefficient which was used in these calculations included twenty-one Stark broadened lines.

RADIATIVE FLUX

Consider the physical geometry consisting of two parallel black plates, with one plate ($y = 0$) having the temperature T_1 and the other ($y = L$) the temperature T_2. The linearized radiative flux for this system has been expressed by Gille and Goody [3] in terms of the modified emissivity

$$\varepsilon(y) = \frac{1}{4\sigma T_1^3} \int_0^\infty \left(\frac{\mathrm{d}e_\omega}{\mathrm{d}T}\right)_{T_1} (1 - \mathrm{e}^{-\kappa_\omega y})\, \mathrm{d}\omega \quad (1)$$

and its first derivative

$$\varepsilon'(y) = \frac{1}{4\sigma T_1^3} \int_0^\infty \left(\frac{\mathrm{d}e_\omega}{\mathrm{d}T}\right)_{T_1} \kappa_\omega\, \mathrm{e}^{-\kappa_\omega y}\, \mathrm{d}\omega. \quad (2)$$

This formulation made use of the exponential approximation

$$E_2(t) \simeq a\,e^{-bt},\ E_3(t) = -\int E_2(t)\,dt \simeq \frac{a}{b}e^{-bt}. \tag{3}$$

Alternatively, Baldwin [5], Wang [4, 7], and Gilles, Cogley and Vincenti [6] have formulated the linearized radiative flux without making use of equation (3). Following Wang [4], a mean value of $\varepsilon(y)$ is defined as

$$\bar{\varepsilon}(y) = \int_0^1 \varepsilon(y/\mu)\,\mu\,d\mu$$

such that from equations (1) and (2)*

$$\bar{\varepsilon}(y) = \frac{1}{4\sigma T_1^3}\int_0^\infty \left(\frac{de_\omega}{dT}\right)_{T_1}[\tfrac{1}{2} - E_3(\kappa_\omega y)]\,d\omega \tag{4}$$

$$\bar{\varepsilon}'(y) = \frac{1}{4\sigma T_1^3}\int_0^\infty \left(\frac{de_\omega}{dT}\right)_{T_1}\kappa_\omega E_2(\kappa_\omega y)\,d\omega. \tag{5}$$

The linearized radiative flux (total over all wave numbers) is in turn [4]

$$q_R = 8\sigma T_1^3(T_1 - T_2)[\tfrac{1}{2} - \bar{\varepsilon}(L-y) + 8\sigma T_1^3\int_0^y [T(y') - T_1]\,\bar{\varepsilon}'(y-y')\,dy' - 8\sigma T_1^3\int_y^L [T(y') - T_1]\,\bar{\varepsilon}'(y'-y)\,dy' \tag{6}$$

where the linearization has been taken about the temperature T_1, and κ_ω thus corresponds to this temperature.

The above equation may now be rephrased in terms of $\varepsilon(y)$ and $\varepsilon'(y)$ through use of the exponential approximation. Upon substituting equation (3) into equations (4) and (5), it is easily shown that equation (6) becomes

$$q_R = \frac{8a}{b}\sigma T_1^3(T_1 - T_2)\{1 - \varepsilon[b(L-y)]\} + 8a\sigma T_1^3\int_0^y [T(y') - T_1]\,\varepsilon'[b(y-y')]\,dy' - 8a\sigma T_1^3\int_y^L [T(y') - T_1]\,\varepsilon'[b(y'-y)]\,dy'. \tag{7}$$

For $b = 2a = 1{\cdot}667$, this coincides with the formulation of Gille and Goody [3], while in the present study values for a and b will be chosen as

$$a = \tfrac{3}{4} \qquad b = \tfrac{3}{2}. \tag{8}$$

In addition to equations (6) and (7), which are general expressions for all optical thicknesses, there also exist the well-known optically thin and optically thick limits. As discussed by Cogley, Vincenti and Gilles [13], and by Wang [4], the linear Planck mean coefficient applies when the radiation is both linearized and optically thin. This mean coefficient is defined as

$$\kappa_{LP} = \frac{\int_0^\infty \kappa_\omega (de_\omega/dT)_{T_1}\,d\omega}{4\sigma T_1^3}. \tag{9}$$

The optically thin expression for the one-dimensional divergence of the radiative flux vector may be obtained by differentiating equation (6), noting from equations (5) and (9) that in the optically thin limit $\bar{\varepsilon}' = \kappa_{LP}$ [since $E_2(\kappa_\omega y) \simeq 1$ for $\kappa_\omega y \ll 1$], and deleting the self absorption terms. The result is

$$-\frac{dq_R}{dy} = 8\sigma T_1^3\kappa_{LP}(T_1 + T_2) - 16\sigma T_1^3\kappa_{LP}T(y). \tag{10}$$

When the radiation is optically thick, the radiative flux is described by the Rosseland equation, which for linearized radiation has the form [13]

$$q_R = -\frac{16\sigma T_1^3}{3\kappa_R}\frac{dT}{dy}, \tag{11}$$

where

$$\frac{1}{\kappa_R} = \frac{1}{4\sigma T_1^3}\int_0^\infty \frac{1}{\kappa_\omega}\left(\frac{de_\omega}{dT}\right)_{T_1}d\omega \tag{12}$$

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* In the nomenclature of Gilles, Cogley and Vincenti [6], $\bar{\varepsilon}(y) = \frac{1}{2} - F_3(y)$ and $\bar{\varepsilon}'(y) = F_2(y)$.

is the conventional Rosseland mean absorption coefficient.

DIFFERENTIAL APPROXIMATION

The differential approximation has seen wide application in gray gas analyses. A nongray extension of the differential approximation has been proposed by Traugott [14] for the emission-controlled limit (cold boundaries), and the basis of this extension was the requirement that the differential approximation reduce to the correct thin and thick limits. Cogley, Vincenti and Gilles [13] have discussed this for linearized radiation. More recently Gilles, Cogley and Vincenti [6] have shown that a differential approximation for linearized radiation can be obtained through approximating $\bar{\varepsilon}'(y)$ by an exponential function in equation (6).

A similar result may be obtained from equation (7) by letting*

$$\varepsilon'(y) \simeq \kappa_1 e^{-\kappa_2 y}, \quad 1 - \varepsilon(y) \simeq \frac{\kappa_1}{\kappa_2} e^{-\kappa_2 y} \quad (13)$$

where κ_1 and κ_2 are as yet unspecified mean absorption coefficients. Upon substituting equation (13) into equation (7), differentiating twice, and making use of equation (8), the differential approximation for q_R is

$$\frac{d^2 q_R}{dy^2} - \tfrac{9}{4}\kappa_2^2 q_R = 12\kappa_1 \sigma T_1^3 \frac{dT}{dy}. \quad (14)$$

The boundary conditions for equation (14) follow by substituting equations (8) and (13) into equation (7), and combining the resulting expression for q_R with its first derivative at each boundary [6]. This yields

$$\left(\frac{dq_R}{dy}\right)_{y=0} - \tfrac{3}{2}\kappa_2 q_R(0) = 12\kappa_1 \sigma T_1^3 [T(0) - T_1] \quad (15a)$$

$$\left(\frac{dq_R}{dy}\right)_{y=L} + \tfrac{3}{2}\kappa_2 q_R(L) = 12\kappa_1 \sigma T_1^3 [T(L) - T_2]. \quad (15b)$$

* A parallel treatment, applicable to the emission-controlled limit, has recently been given by Traugott [16]

Three pairs of choices for κ_1 and κ_2 have been suggested by Gilles, Cogley and Vincenti [6], and these are*

$$\kappa_1 = \kappa_2 = \kappa_{LP} \quad (16)$$

$$\kappa_1 = \kappa_2 = \kappa_R \quad (17)$$

$$\kappa_1 = \kappa_{LP}, \kappa_2 = (\kappa_{LP}\kappa_R)^{\frac{1}{2}}. \quad (18)$$

The use of equations (16) or (17) corresponds to the gray gas assumption applied to equation (7), with either κ_{LP} or κ_R being employed as the mean absorption coefficient. As discussed in reference [6], the use of equation (18) in equation (14) yields the Rosseland equation in the optically thick limit, while the correct form for dq_R/dy is obtained in the optically thin limit.† However, equation (18) does not produce the correct expression for q_R in the optically thin limit. This may easily be shown by substituting equations (8) and (13) into equation (7), and in the optically thin limit this reduces to

$$q_R = 4\sigma\left(\frac{\kappa_1}{\kappa_2}\right) T_1^4 (T - T_2), \quad (19)$$

which is correct only for $\kappa_1 = \kappa_2$. To give a specific example, for a hydrogen plasma with $T^1 = 10{\cdot}000$°K and $P = 37{\cdot}5$ atm, then, according to equation (18), $\kappa_1/\kappa_2 = 44$ [12].

RADIATIVE TRANSFER ANALYSES

The radiative flux expressions discussed in the two preceding sections will now be applied to the illustrative problem of a hydrogen plasma bounded by two parallel black plates. Within

* Actually the proposed κ_1 and κ_2 results of reference [6] correspond, in the present nomenclature, to different values of a and b for each pair of κ_1 and κ_2. Specifically, $a = 1$ and $b = 2$ for equation (16), $a = \frac{3}{4}$ and $b = \frac{3}{2}$ for equation (17), and $a = 1$ and $b = \sqrt{3}$ for equation (18). In the present investigation, $a = \frac{3}{4}$ and $b = \frac{3}{2}$ will be used throughout.

† Due to the present choice for a and b, equation (18) will actually yield a result for the optically thin limit which is $\frac{3}{4}$ times the exact expression given by equation (10) This limiting form is obtained by noting that the second term on the left side of equation (14) may be deleted under optically thin conditions, such that equation (14) can be integrated directly The constant of integration is evaluated from the sum of equations (15a) and (15b), with $q_R(0) = q_R(L)$ for optically thin radiation

the plasma there is a uniform heat source per unit volume, Q, and the two plates have the same uniform temperature T_1 (i.e. $T_2 = T_1$).

With respect to the integral formulation for the radiative flux, equation (7) will be employed rather than equation (6). It should be recalled that equation (7) is an approximate form of equation (6), and that this approximation refers to the use of equation (3). The application of both equations (6) and (7) to i.r. radiative transfer has recently been considered by Greif and Habib [11], from which it is found that the essential characteristics of the integral formulation are retained by equation (7). As previously discussed, results for $\varepsilon(y)$, $\varepsilon'(y)$, κ_{LP} and κ_R are given in reference [12] for a hydrogen plasma, and the following analyses will employ these quantities.

In order to clearly illustrate the radiative transfer process within the plasma, solutions will first be obtained neglecting thermal conduction as an energy transport mechanism. Conduction will then be included, so that the relative importance of conduction and radiation as transport mechanisms may be determined.

Conduction neglected

For uniform heat generation, and with conduction neglected, the statement of conservation of energy is that $dq_R/dy = Q$. Since the problem is symmetric, integration of this yields

$$Q\left(y - \frac{L}{2}\right) = q_R. \tag{20}$$

Upon letting

$$\theta = \frac{4\sigma T_1^3(T - T_1)}{QL}$$

then equations (7), (8) and (20) combine to yield

$$\frac{u}{u_0} - \frac{1}{2} = \frac{3}{2}\int_0^u \theta(u')\,\varepsilon'[\tfrac{3}{2}(u - u')]\,du' - \frac{3}{2}\int_u^{u_0} \theta(u')\,\varepsilon'[\tfrac{3}{2}(u' - u)]\,du', \tag{21}$$

where $\varepsilon'(u)$ denotes the derivative of ε with respect to u, and $u = P_H y$, where P_H is the partial pressure of atomic hydrogen. Equation (21) constitutes an integral equation describing the dimensionless temperature profile, $\theta(u)$, and the kernel of this equation, $\varepsilon'(u)$, is tabulated in reference [12] for hydrogen.

The optically thin solution may be obtained directly from equation (21). Analogous to the discussion preceding equation (10), optically thin radiation corresponds to

$$\varepsilon'(u) = \frac{\kappa_{LP}}{P_H}$$

and the optically thin solution of equation (21) is found to be

$$\theta(u) = \frac{1}{3L\kappa_P}. \tag{22}$$

As previously discussed, this result is in error by a factor of $\frac{4}{3}$ due to the present choice for a and b. Equation (22) is, however, consistent with equation (21) as well as with equations (14) and (15).

The optically thick solution follows from equations (11) and (20) as

$$\theta(u) = \tfrac{3}{8}\kappa_R L\left(\frac{u}{u_0} - \frac{u^2}{u_0^2}\right), \tag{23}$$

while combining equations (14), (15) and (20) yields

$$\theta(u) = \frac{3}{8}\left(\frac{\kappa_2}{\kappa_1}\right)\kappa_2 L\left(\frac{u}{u_0} - \frac{u^2}{u_0^2}\right) + \frac{1}{3\kappa_1 L}(1 + \tfrac{3}{4}\kappa_2 L) \tag{24}$$

for the solution corresponding to the differential approximation.

Equation (21) has been solved numerically for $\theta(u)$ employing the same method used in [8], and for the sake of brevity only the centerline temperature will be presented; that is, $T_c = T(L/2)$. This is represented by the solid curve in Fig. 1 for $T_1 = 10\,000$°K and an electron density of 10^{17} cm^{-3}, which corresponds to a total pressure of 37·5 atm. Also shown are results utilizing continuum radiation only, and this solution was obtained from equation (21) by employing values of $\varepsilon'(u)$ from reference [12] for which lines were not included in the calculation. It may be seen that the importance of line radiation is relatively small for the conditions of Fig. 1, and the reason for this is that the lines are nearly opaque and thus do not contribute significantly to the radiative transfer process. In addition, for the conditions of Fig. 1 the Lyman continuum is also nearly opaque. Consequently the radiative transport process illustrated in Fig. 1 is due primarily to continuum radiation, with the Lyman continuum excluded. One may note the relative invariance of the solid curve with L for the smaller L values, and this is indicative of optically thin radiation.* The implication is clearly that, while the lines and the Lyman continuum are nearly opaque, the remaining continuum radiation is nearly optically thin for small L values.

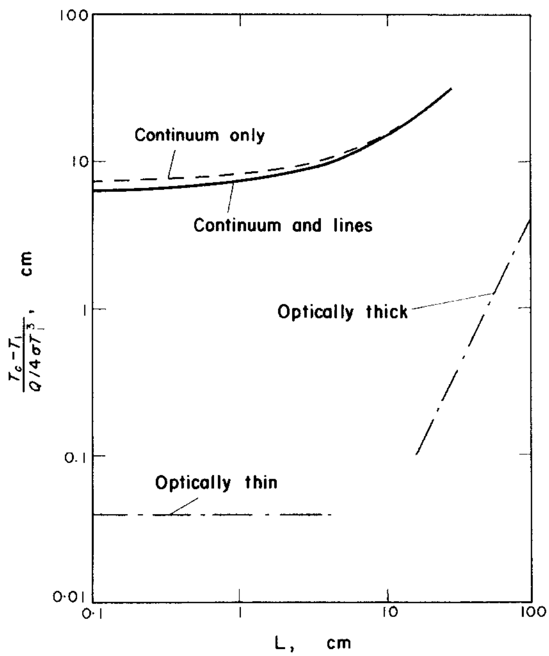

FIG. 1. Centerline temperature results with conduction neglected, $T_1 = 10\,000°K$, $N_e = 10^{17}\ cm^{-3}$ ($P = 37·5$ atm).

* See equation (22) and note that the ordinate of Fig. 1 corresponds to $L\theta(u_0/2)$.

The optically thin and optically thick solutions, corresponding to equations (22) and (23), respectively, are also illustrated in Fig. 1. Since the lines are nearly opaque even for $L = 0·1$ cm, it is obvious that excessively small values of L would be required for the solution of equations (21) to approach the optically thin limit (i.e. for *both* line and continuum radiation to be optically thin). Conversely, extremely large L values are necessary for continuum radiation to become optically thick, which must occur in order for the solution of equation (21) to approach the optically thick limit.

Similar comparisons are shown in Fig. 2 for

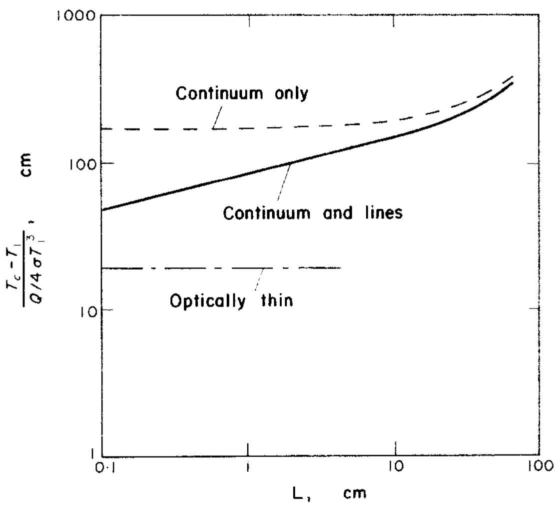

FIG. 2. Centerline temperature results with conduction neglected, $T_1 = 20\,000°K$, $N_e = 10^{17}\ cm^{-3}$ ($P = 0·57$ atm).

$T_1 = 20\,000°K$ and the same electron density ($10^{17}\ cm^{-3}$), which in this case corresponds to a total pressure of 0·57 atm. Here the line contribution is much more pronounced, which results from a combination of the lower pressure and higher temperature. This gives rise to lower spectral optical thicknesses for the lines, relative to the conditions of Fig. 1, such that, except for very large values of L, the lines are not opaque and thus contribute to the radiative transfer process. Note that the optically thin solution is a more realistic limit than for the previous comparison, although extremely small L values would still be required in order to approach optically thin radiation. On the other hand, the optically thick solution is even less useful than for the conditions of Fig. 1, and it is not possible to include it in Fig. 2. To give an example, for $L = 100$ cm the optically thick limit yields an ordinate value of 0·081, which is roughly four orders of magnitude less than what would be predicted by the solution of equation (21).

An additional comparison is shown in Fig. 3 for $T_1 = 10\,000°K$ and an electron density of 10^{16} cm^{-3}, which corresponds to a pressure of 0·43 atm. As would be expected, the importance of line radiation is more pronounced than in Fig. 1, since the pressure is substantially less than for Fig. 1. On the other hand, a comparison of Figs. 2 and 3 illustrates the effect of temperature upon the line contributions, since the two pressures are relatively close to one another. While the line contribution steadily diminishes with increasing L in Fig. 2, this is not the case for the lower temperature results of Fig. 3.

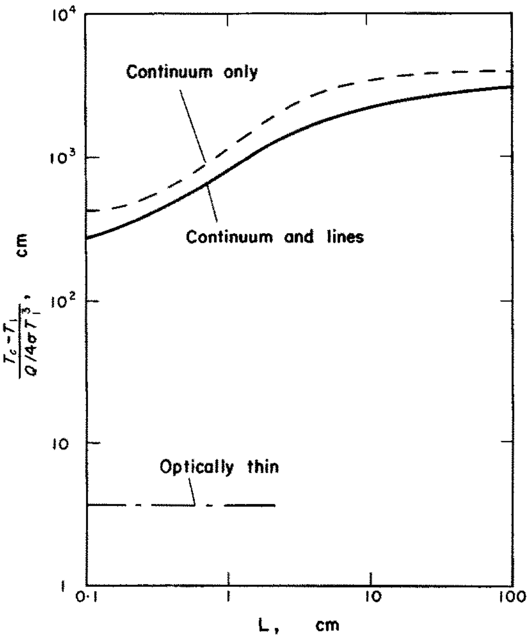

FIG. 3. Centerline temperature results with conduction neglected, $T_1 = 10\,000°K$, $N_e = 10^{16}$ cm^{-3} ($P = 0·43$ atm).

Evidently, for the conditions of Fig. 3, the line contribution for small L values is due mainly to a few strong lines. As L increases, these lines become opaque, while weaker lines become increasingly important. Eventually, for very large values of L, the line contribution will become negligible.

As in the previous figure, the radiative transfer process for the conditions of Fig. 3 is far from approaching optically thick radiation, with the optically thick limit underestimating the ordinate value by roughly five orders of magnitude for $L = 100$ cm.

The somewhat different shape of the solid curve in Fig. 3, as opposed to Figs. 1 and 2, can best be explained by considering the dashed curve representing continuum radiation only. For small values of L, this curve approaches a constant value indicative of optically thin radiation, and for the conditions of Fig. 3 this corresponds to the entire continuum being optically thin. The dashed curve also indicates optically thin radiation for large L values, and this is a consequence of the Lyman continuum becoming opaque while the remaining continuum is still optically thin. This transition of the Lyman continuum from optically thin to opaque in turn dictates the shape of the solid curve representing both line and continuum radiation.

Results employing the differential approximation, equation (24), are compared in Figs. 4 and 5 with the solution of equation (21). Unfortunately, the differential approximation, as presently employed, does not appear to be a particularly useful approximation. The possible failure of the differential approximation has been predicted by Gilles, Cogley and Vincenti [6], and they suggest that it may be necessary to replace equation (13) by a sum of exponentials in order "to represent a gas that is simultaneously thick and thin in different parts of the spectrum". Indeed, in the present situation line radiation generally corresponds to large optical thicknesses, while continuum radiation tends toward much smaller optical thicknesses. Similar

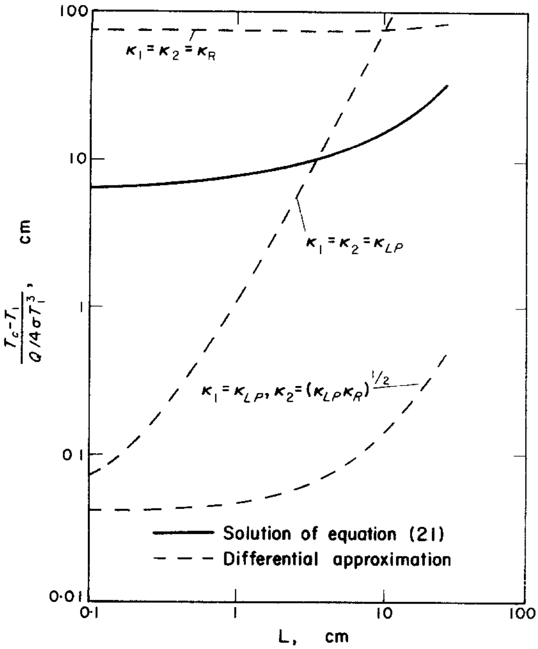

FIG. 4. Centerline temperature results with conduction neglected, $T_1 = 10\,000°\text{K}$, $N_e = 10^{17}\ \text{cm}^{-3}$ ($P = 37{\cdot}5$ atm).

conclusions have been reached by Traugott [16] for the emission-controlled limit.

Conduction included

When conduction is included as a transport

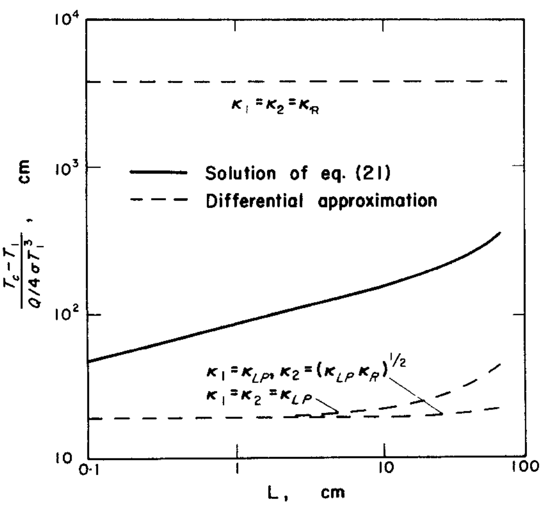

FIG. 5. Centerline temperature results with conduction neglected, $T_1 = 20\,000°\text{K}$, $N_e = 10^{17}\ \text{cm}^{-3}$ ($P = 0{\cdot}57$ atm).

mechanism within the gas, the statement of conservation of energy becomes

$$\lambda \frac{\mathrm{d}T}{\mathrm{d}y} + Q\left(y - \frac{L}{2}\right) = q_R. \tag{25}$$

In this case a dimensionless temperature difference will be defined as

$$\phi = \frac{T - T_1}{QL^2/\lambda}$$

and equations (7), (8) and (25) combine to yield

$$u_0 \frac{\mathrm{d}\phi}{\mathrm{d}u} + \left(\frac{u}{u_0} - \frac{1}{2}\right) = \left(\frac{6\sigma T_1^3 L}{\lambda}\right)\left\{\int_0^u \phi(u')\,\varepsilon'[\tfrac{3}{2}(u - u')]\,\mathrm{d}u - \int_u^{u_0} \phi(u')\,\varepsilon'[\tfrac{3}{2}(u' - u)]\,\mathrm{d}u'\right\}. \tag{26}$$

Since the presence of conduction implies continuity of temperature at the boundaries, the boundary condition for this equation is

$$\phi(0) = 0. \tag{27}$$

Equation (26), subject to equation (27), has been solved numerically employing thermal conductivity values of Yos [15]. Note that for negligible radiation transfer

$$\phi = \frac{1}{2}\left(\frac{u}{u_0} - \frac{u^2}{u_0^2}\right). \tag{28}$$

Dimensionless centerline temperatures, as obtained from the solution of equation (26), are illustrated in Figs. 6 and 7. Since the centerline temperature for pure conduction follows from equation (28) to be

$$\frac{T_c - T_1}{QL^2/\lambda} = 0{\cdot}125,$$

then Figs. 6 and 7 serve to illustrate the effect of radiative transfer upon the temperature profile within the gas. As would be expected, the importance of radiation becomes more pronounced as the plate spacing is increased.

The effect of pressure, as illustrated in Fig. 6, is in accord with Figs. 1 and 3, since for the conditions considered an increase in pressure results in an increase in radiative transport within the plasma. The influence of line radiation, which is illustrated in Fig. 6 for 37·5 atm as well as in Fig. 7, also agrees with the previous results of Figs. 1 and 2.

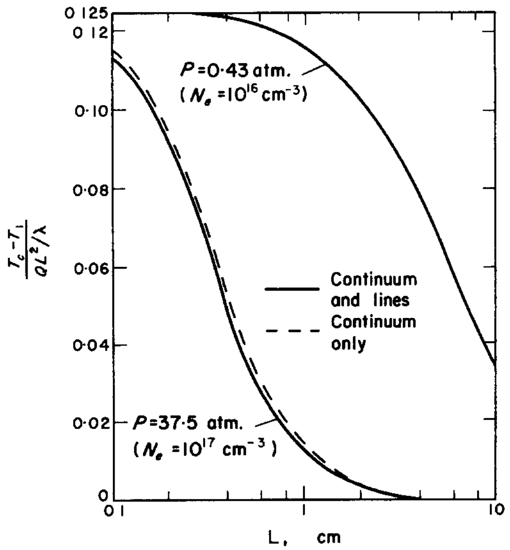

FIG. 6. Centerline temperature results with conduction included, $T_1 = 10\,000°K$.

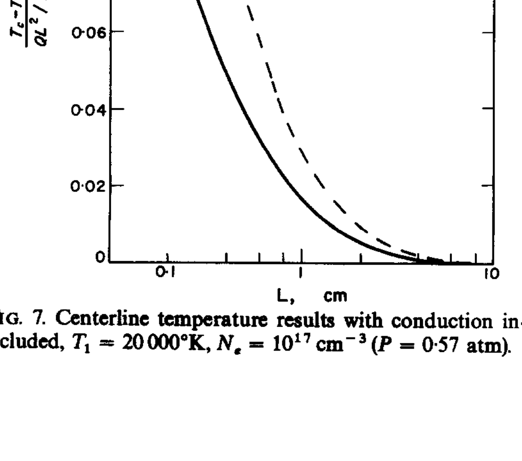

FIG. 7. Centerline temperature results with conduction included, $T_1 = 20\,000°K$, $N_e = 10^{17}\ cm^{-3}$ ($P = 0·57$ atm).

What is perhaps most surprising about the information presented in Figs. 6 and 7 is the large role radiation plays as an internal transport mechanism. To give an example, for a temperature of 20 000°K and a pressure of 0·57 atm, the centerline temperature difference is reduced by approximately one half, relative to pure conduction, for $L = 0·4$ cm, which implies that for these conditions radiation and conduction are equally important energy transport mechanisms. For the same temperature and pressure, but with $L = 1$ cm, the centerline temperature difference is reduced by an order of magnitude, and thus radiation completely dominates over conduction.

DISCUSSION OF RESULTS

Although the results of the previous section are quite limited in extent, it is, nevertheless, clear that the practical utility of either the optically thin or optically thick limits is very restricted, and this is due to the large range of optical thicknesses that occur under actual conditions. It should be emphasized, however, that the preceding results characterize a situation for which the net radiative transfer is solely between the gas and the bounding surfaces; that is, there is no net radiative transfer between surfaces. In this regard, it is of interest to consider briefly the opposite extreme of a problem for which the net radiative transfer is strictly between the surfaces. Such a situation is that of radiative equilibrium for a hydrogen plasma bounded by two parallel surfaces having different temperatures T_1 and T_2.

The state of radiative equilibrium is described by $dq_R/dy = 0$; that is, there is no net energy transfer to or from the gas, such that $q_R =$ constant. Upon defining

$$\bar{Q} = \frac{q_R}{4\sigma T_1^3(T_1 - T_2)}, \qquad \eta = \frac{T - T_1}{T_2 - T_1},$$

then equations (7) and (8) yield

$$\bar{Q} = 1 - \varepsilon[\tfrac{3}{2}(u_0 - u)] - \tfrac{3}{2}\int_0^u \eta(u')\varepsilon'[\tfrac{3}{2}u - u')]\,du' + \tfrac{3}{2}\int_u^{u_0} \eta(u')\,\varepsilon'[\tfrac{3}{2}(u' - u)]\,du'. \quad (29)$$

This together with the radiative equilibrium condition

$$\bar{Q} = \text{const.} \quad (30)$$

completely describes both $\eta(u)$ and $\bar{Q}$. This system of equations has been solved numerically, and the results for the dimensionless heat flux between surfaces, $\bar{Q}$, are illustrated in Fig. 8. In this case the optically thin limit coincides with the result for a transparent gas, for which $\bar{Q} = 1$.

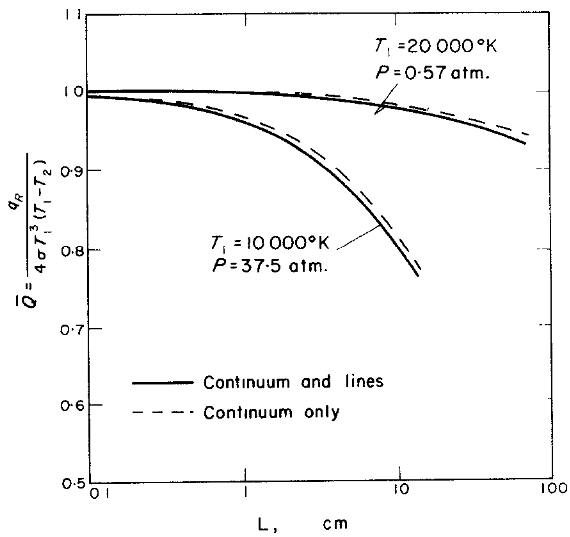

FIG 8. Heat flux results for radiative equilibrium, $N_e = 10^{17}$ cm^{-3}

Consider as an example $T_1 = 10\,000$°K, $P = 37·5$ atm, and $L = 1$ cm. From Fig. 8, $\bar{Q} = 0·96$, and this is sufficiently close to unity so that the radiative equilibrium problem might be regarded as optically thin. On the other hand, with reference to Fig. 1 the radiative transfer process within the plasma is far from being optically thin for the uniform heat generation problem.

This difference in interpretation is due mainly to the role played by line radiation. With reference to Fig. 1, it has previously been discussed that the lines are essentially opaque, while for L values less than approximately 1 cm, continuum radiation (with the exception of the Lyman continuum) is close to being optically thin. Thus, the departure from optically thin radiation in Fig. 1 is primarily a consequence of the lines being nonthin. Conversely, the fact that for radiative equilibrium the opaque lines do not substantially reduce $\bar{Q}$ is a result of line radiation encompassing only a small fraction of the radiation spectrum. The optically thin continuum radiation, however, encompasses most of the spectrum, and, for the conditions previously stated, since this radiation is nearly optically thin, the resulting value for $\bar{Q}$ will in turn be close to the optically thin limit.

In effect, line radiation appears to have little influence upon radiative transfer through the plasma, as is further shown in Fig. 8 for $T_1 = 20\,000$ K and $P = 0·57$ atm, while it may or may not play a significant role as an energy transport mechanism within the plasma, depending upon whether the line radiation is or is not opaque. Again referring to the above conditions of $T_1 = 20\,000$°K and $P = 0·57$ atm, Figs. 2 and 7 illustrate that line radiation can indeed be an important energy transport mechanism within the plasma.

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TRANSPORT D'ÉNERGIE PAR RAYONNEMENT À L'INTÉRIEUR D'UN PLASMA D'HYDROGÈNE

Résumé—L'objet de cette étude est d'examiner diverses méthodes d'analyse du transport d'énergie par rayonnement dans un plasma. Dans ce but, un système physique très simple est choisi, consistant en un plasma d'hydrogène limité par deux plaues noires parallèles, et à l'intérieur une source uniforme de chaleur par unité de volume. Des résultats restreints sont présentés aussi pour un plasma d'hydrogène en équilibre de rayonnement.

Puisque le rayonnement par raies est caractérisé généralement par des épaisseurs optiques relativement grandes, tandis que le rayonnement continu correspond à des épaisseurs optiques beaucoup plus petites, on trouve que les limites optiquement mince et optiquement épaisse ont peu d'utilité pratique, en ce que les rayonnements par raies et continu seront rarement soit fins soit épais simultanément. Cette grande variation dans l'épaisseur optique spectacle apparaît en outre limiter l'utilité d'une approximation différentielle non grise, grâce à laquelle le flux rayonnant est décrit à l'aide d'une équation différentielle du second ordre.

On montre que le rayonnement par raies peut avoir un effet sensible sur le transport par énergie de rayonnement dans le plasma pour des pressions modérées. A des pressions plus élevées, cependant, les raies deviennent opaques et ne contribuent plus au processus de transport par rayonnement. Dans l'un ou l'autre cas, les raies semblent avoir peu d'influence sur le rayonnement passant à travers le plasma (en opposition avec le transport par rayonnement à l'intérieur du plasma), puisque les raies ne concernent seulement qu'une petite fraction du spectre.

ENERGIETRANSPORT DURCH STRAHLUNG IN EINEM WASSERSTOFF-PLASMA

Zusammenfassung—Zweck dieser Untersuchung ist es, verschiedene Berechnungsmethoden für die Wärmestrahlung in einem Plasma zu studieren. Dazu wird ein sehr einfaches physikalisches System gewählt:

Ein Wasserstoffplasma, das durch zwei parallele schwarze Platten begrenzt ist, und in dem pro Volumeneinheit eine konstante Wärmemenge produziert wird. Begrenzt gültige Ergebnisse werden auch für ein sich im Strahlungsgleichgewicht befindliches Wasserstoffplasma angegeben.

Da "Linien"-Strahlung im Allgemeinen durch relativ grosse optische Dicken charakterisiert ist, während bei kontinuierlicher Strahlung meist weit kleinere Dicken vorliegen, ergibt sich, dass die Lösung für die optich dünnen und optisch dicken Grenzfälle wenig praktischen Wert besitzen, weil "Linien"-Strahlung und kontinuierliche Strahlung zusammen selten gleichzeitig optisch dünn oder sein werden. Die grossen Unterschiede in den spektralen optischen Dicken scheinen ferner die Anwendbarkeit einer Näherung für nicht-graue Strahlung, bei der der Strahlungsstrom i durch eine Differentialgleichung 2. Ordnung beschrieben wird, zu begrenzen.

Es wird gezeigt, dass die "Linien"-Strahlung bei mässigen Drücken einen beträchtlichen Einfluss auf den Energietransport innerhalb des Plasmas haben kann. Bei höheren Drücken werden die "Linien" allerdings undurchlässig und die von ihnen ausgehende Strahlung trägt nichts mehr zum Energietransport bei. In jedem Fall scheinen die Linien wenig Einfluss auf Strahlung zu haben die durch das Plasma hindurchgeht (im Gegensatz zum Strahlungstransport innerhalb des Plasmas), da Sie nur einen kleinen Bruchteil des Spektrums umfassen.

ЛУЧИСТЫЙ ЭНЕРГООБМЕН В ВОДОРОДНОЙ ПЛАЗМЕ

Аннотация—Целью данной работы было изучение различных методов анализа переноса лучистой энергии в плазме. Для етой цели была выбрана простая физическая система, состоящая из вгдгргдной плазмы, ораниченной двумя параллельными черными пластинами, а внутри плазмы имелся однородный источник тепла. Представлены также некоторые результаты для водородной плазмы при лучистом равновесии.

Поскольку линейное излучение обычно характеризуется относительно большими плотностями, тогда как непрерывному излучению соответствуют много меньшие оптические плотности, найдено, что малые и большие пределы оптической плотности практически мало полезны, так как линейное излучение и непрерывное излучение редко всгречаются одновременно малыми или болъшими. Оказывается, что такое большое изменение спектральной оптической плотности ограничивает использование несерой дифференциальной аппроксимации, с использованием кокорой лучистый поток описывается дифференциальным уравнением второго порядка.

Показано, что линейное излучение может иметь значительное влияние на перенос лучистой энергии внутри плазмы для умеренных давлений. Однако, при более высоких давлениях линии становятся непрозрачными и не оказывают влияния на процесс лучистого переноса. В любом случае оказывается, что линии мало влияют на проходящее через плазму излучение (в противоположность лучистому переносу внутри плазмы), поскольку линии охватывают только небольшую долю спектра.